ESC103 Unit 18

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1 Higher Order Systems

More than one differential equation is considered higher order:

$$\frac{dx(t)}{dt} = x'(t) = ax(t) + by(t)$$

$$\frac{dy(t)}{dt} = y'(t) = cx(t) + dy(t)$$

a, b, c, d are scalars. This type of system is considered a "linear differential equations with constant coefficients".
Let's say we're given:

$$x(t = 0) = x(0)$$

$$y(t = 0) = y(0)$$

We want to find values of x and y going forward in time, that satisfy the conditions.
As it turns out, it's useful to formulate this using vectors and matrices:
Let:

$$Z = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\therefore Z' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$Z' = AZ \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and:

$$Z = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

Now we can apply Eulers method, algorithm:

$$t_{n+1} = t_n + \Delta t$$

Update entire vector as follows:

$$Z_n + 1 = Z_N + \Delta t A Z_n$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \Delta t \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$x_{n+1} = x_n + \Delta t(ax_n + by_n)$$

$$y_{n+1} = y_n + \Delta t(cx_n + dy_n)$$

These are now called difference equations **NOT** differential equations.

2 Improved Euler's Method Matlab Psudocode:

Exactly as will be coded in matlab:
First we will have Z_n defined somehwere up here, DON'T MODIFY THAT until we are done with the last two lines:

$$Z_{n+1} = Z_n + \Delta t A Z_n$$

$$Z_{n+1} = Z_n + \frac{\Delta t}{2}\left(AZ_n + AZ_{n+1}\right)$$

Second line overwrites first line, so both these lines really only compute the single next update combined. Now it's okay for use to save Z_{n+1} into Z_n.

General Form of 2nd order Differential Equations:

$$\frac{d^2y(t)}{dt^2} = y'(t) = f(t, y(t), y'(t))$$

and given:

$$y(t = 0) = y(0)$$
$$y'(t = 0) = y'(0)$$

Notice how we need TWO initial conditions instead of one, in general, the order of the differential equation is the number of initial conditions that need to be known in order to solve.

Let:

$$Z = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$
$$Z' = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$$

We also need initial conditions:

$$Z_0 = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

Example:

$$y''(t) = -y(t)$$
$$y(0) = 1$$
$$y'(0) = 0$$
$$Z' = AZ$$
$$\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

The A matrix captures the second order equation relationships.

$$Z_0 = \begin{bmatrix} y(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Euler's Method:

We will now use the update equations in order to determine Z_{n+1}

$$Z_{n+1} = Z_n + \Delta t A Z_n$$

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \Delta t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

For Matlab purposes we can leave it in the above matrix form, but let's write it out for us to see:

$$y_{n+1} = y_n + \Delta t y'_n$$

$$y'_{n+1} = y'_n + \Delta t(-y_n)$$

What we're solving for at every time step: y, and the first derivative of y. So if solving F = ma for example, we would get position AND velocity out of it.(Maybe I misheard the last details)

$$Z' = AZ$$

$$Z' = \begin{bmatrix} 0 \\ 50.82 \\ -60.92 \\ 60.92 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 50.82 \\ -60.92 \\ 60.92 \end{bmatrix} = \begin{bmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x'_1(t) \\ x_2(t) \\ x'_2(t)' \end{bmatrix}$$